



TITLE:

# A Sum related to the Eratosthenes Sieve

AUTHOR(S):

Motohashi, Yoichi

---

CITATION:

Motohashi, Yoichi. A Sum related to the Eratosthenes Sieve. 数理解析研究所講究録 1984, 537: 29-34

ISSUE DATE:

1984-10

URL:

<http://hdl.handle.net/2433/98698>

RIGHT:

# A Sum related to the Eratosthenes Sieve

By Yoichi Motohashi

1. In their recent work "Sur une somme liée à la fonction de Möbius" Dress, Iwaniec and Tenenbaum proved the asymptotic formula

$$(1) \quad \sum_{d_1, d_2 \leq Z} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} = (1+o(1)) \frac{1}{\pi} \int_0^\infty \prod_p \left(1 - \frac{1}{p}\right) \left(1 + \frac{1}{p} (1 - p^{-it})^2 - 1\right) \frac{dt}{t^2}$$

as  $Z$  tends to infinity. This has some interest in its connection with the Eratosthenes sieve through the relation

$$\sum_{n \leq X} \left( \sum_{\substack{d|n \\ d \leq Z}} \mu(d) \right)^2 = X \sum_{d_1, d_2 \leq Z} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2]} + O(Z^2).$$

To show (1) they made use of a real functional method. But, as I remarked in a letter to Dress, a routine application of Perron's inversion formula to the function

$$\sum_{d_1, d_2} \frac{\mu(d_1)\mu(d_2)}{[d_1, d_2] d_1^{s_1} d_2^{s_2}} \quad (Re(s_j) > 0, j=1, 2)$$

yields (1) with an explicit error of the order of  $\exp(-c(\log Z / \log \log Z)^{1/3})$

Here, employing the same argument I will give a brief proof of a little more difficult

## THEOREM

$$(2) \quad \sum_{d_1, d_2, d_3, d_4 \leq Z} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4)}{[d_1, d_2, d_3, d_4]} = C_0 \log^2 Z + O(\log Z),$$

$$C_0 = \frac{3}{2\pi} \int_0^\infty \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{1}{p} (4|1-p^{-it}|^2 - |1-p^{-2it}|^2 - 1)\right) \frac{dt}{t^4}.$$

Remark

(i) Our sum comes from

$$\sum_{n \leq X} \left( \sum_{\substack{d|n \\ d \leq Z}} \mu(d) \right)^4.$$

(ii) As is apparent from our proof the right side of (2) can be put more precisely in the form  $c_0(\log Z)^2 + c_1 \log Z + c_2 + O(\exp(-c(\frac{\log Z}{\log \log Z})^{\frac{1}{3}}))$ , though the explicit computation of  $c_1$  and  $c_2$  is quite tedious.

2. Let  $s_j$  ( $1 \leq j \leq 4$ ) be complex variables, and put

$$I_j = \{s_j; \operatorname{Re}(s_j) = j\eta, |\operatorname{Im}(s_j)| \leq jZ^2\} \quad (1 \leq j \leq 3),$$

$$I_4 = \{s_4; \operatorname{Re}(s_4) = 7\eta, |\operatorname{Im}(s_4)| \leq 7Z^2\},$$

where  $\eta = c(\log Z)^{\frac{2}{3}}(\log \log Z)^{-\frac{1}{3}}$  (cf. Vinogradov's zero-free region).

Also we assume that  $Z = [Z] + 1/2$ , without losing any generality.

Then, applying Perron's inversion formula four times we see that the sum to be estimated is equal to

$$\begin{aligned} & \frac{1}{(2\pi i)^4} \iiint_{\substack{s_j \in I_j \\ (1 \leq j \leq 4)}} \sum_{d_1, d_2, d_3, d_4=1}^{\infty} \frac{\mu(d_1)\mu(d_2)\mu(d_3)\mu(d_4)}{[d_1, d_2, d_3, d_4] d_1^{s_1} d_2^{s_2} d_3^{s_3} d_4^{s_4}} \frac{Z^{s_1+s_2+s_3+s_4}}{s_1 s_2 s_3 s_4} ds_1 ds_2 ds_3 ds_4 \\ & \quad + O(Z^{-\frac{1}{2}}) \\ & = J + O(Z^{-\frac{1}{2}}), \end{aligned}$$

say. The sum in the integrand is equal to

$$\begin{aligned} & \prod_p \left[ 1 + \frac{1}{p} \sum_{l=1}^4 (-1)^l \sum_{1 \leq j_1 < \dots < j_l \leq 4} p^{-s_{j_1} - \dots - s_{j_l}} \right] \\ & = \prod_{l=1}^4 \prod_{1 \leq j_1 < \dots < j_l \leq 4} \zeta(1+s_{j_1} + \dots + s_{j_l})^{(-1)^l} W(s_1, s_2, s_3, s_4); \end{aligned}$$

$W(s_1, s_2, s_3, s_4)$  is regular and bounded in the region  $\operatorname{Re}(s_j) > -c$  ( $1 \leq j \leq 4, c > 0$ ).

3. Shifting the line of integration  $I_4$  of the variable  $s_4$  to  $\{ \operatorname{Re}(s_4) = -7\eta, |\operatorname{Im}(s_4)| \leq 7Z^2 \}$ , we encounter singularities only at  $s_4 = -s_3, -s_2, -s_1, -(s_3 + s_2 + s_1)$ , which are all simple poles. Then computing the residues we get

$$J =$$

$$- \frac{1}{(2\pi i)^3} \iiint_{\substack{s_j \in I_j \\ (1 \leq j \leq 3)}} \frac{\zeta(1+s_1+s_2)^2 \zeta(1+s_1+s_3) \zeta(1+s_1-s_3) \zeta(1+s_2+s_3) \zeta(1+s_2-s_3) W(s_1, s_2, s_3, -s_3) Z^{s_1+s_2}}{\zeta(1+s_1)^2 \zeta(1+s_2)^2 \zeta(1+s_3) \zeta(1-s_3) \zeta(1+s_1+s_2+s_3) \zeta(1+s_1+s_2-s_3) s_1 s_2 s_3^2} ds_1 ds_2 ds_3$$

$$- \frac{1}{(2\pi i)^3} \iiint_{\substack{s_j \in I_j \\ (1 \leq j \leq 3)}} \frac{\zeta(1+s_1+s_2) \zeta(1+s_1+s_3)^2 \zeta(1+s_1-s_2) \zeta(1+s_2+s_3) \zeta(1+s_3-s_2) W(s_1, s_2, s_3, -s_2) Z^{s_1+s_3}}{\zeta(1+s_1)^2 \zeta(1+s_2) \zeta(1+s_3)^2 \zeta(1-s_2) \zeta(1+s_1+s_2+s_3) \zeta(1+s_1+s_3-s_2) s_1 s_2^2 s_3} ds_1 ds_2 ds_3$$

$$- \frac{1}{(2\pi i)^3} \iiint_{\substack{s_j \in I_j \\ (1 \leq j \leq 3)}} \frac{\zeta(1+s_1+s_2) \zeta(1+s_1+s_3) \zeta(1+s_2+s_3)^2 \zeta(1+s_2-s_1) \zeta(1+s_3-s_1) W(s_1, s_2, s_3, -s_1) Z^{s_2+s_3}}{\zeta(1+s_1) \zeta(1+s_2)^2 \zeta(1+s_3)^2 \zeta(1-s_1) \zeta(1+s_1+s_2+s_3) \zeta(1+s_2+s_3-s_1) s_1^2 s_2 s_3} ds_1 ds_2 ds_3$$

$$+ O(1)$$

$$= J_1 + J_2 + J_3 + O(1), \text{ say.}$$

Next, in the integral  $J_1$  we shift the line of integration  $I_2$  of the variable  $s_2$  to the line  $I_2^* = \{ s_2; \operatorname{Re}(s_2) = -2\eta, |\operatorname{Im}(s_2)| \leq 2Z^2 \}$ .

The singularity to be accounted is only at  $s_2 = -s_1$  which is a double pole, and a simple estimation gives

$$J_1 = O(\log Z)$$

As for the integral  $J_2$  we shift the line of integration  $I_3$  of the variable  $s_3$  to the line  $\{s_3; \operatorname{Re}(s_3) = -3\gamma, |\operatorname{Im}(s_3)| \leq 3Z^2\}$ . Then we encounter singularities only at  $s_3 = s_2, -s_2, -s_1$ . The first two are simple poles, and the last is a double pole with the residue of the order of  $\log Z$ . Also it is easy to see that the residue at  $s_3 = -s_2$  is  $O(Z^{-\epsilon})$ . Hence we have

$$J_2 = -\frac{1}{(2\pi i)^2} \iint_{s_1 \in I_1, s_2 \in I_2} \frac{\zeta(1+s_1+s_2)^3 \zeta(1+s_1-s_2) \zeta(1+2s_2) W(s_1, s_2, s_2, -s_2) Z^{s_1+s_2}}{\zeta(1+s_1)^3 \zeta(1+s_2)^3 \zeta(1-s_2) \zeta(1+s_1+2s_2) s_1 s_2^3} ds_1 ds_2 \\ + O(\log Z)$$

$$= J_{21} + O(\log Z), \text{ say.}$$

In much the same way we have

$$J_3 = -\frac{1}{(2\pi i)^2} \iint_{s_1 \in I_1, s_2 \in I_2} \frac{\zeta(1+s_1+s_2)^3 \zeta(1+s_2-s_1) \zeta(1+2s_1) W(s_1, s_2, s_1, -s_1) Z^{s_1+s_2}}{\zeta(1+s_1)^3 \zeta(1+s_2)^3 \zeta(1-s_1) \zeta(1+2s_1+s_2) s_1^3 s_2} ds_1 ds_2 \\ + \frac{1}{(2\pi i)^2} \iint_{s_1 \in I_1, s_2 \in I_2} \frac{\zeta(1+s_1+s_2) \zeta(1+s_2-s_1)^3 \zeta(1-2s_1) W(s_1, s_2, -s_1, -s_1) Z^{s_2-s_1}}{\zeta(1+s_1) \zeta(1+s_2)^3 \zeta(1-s_1)^3 \zeta(1-2s_1+s_2) s_1^3 s_2} ds_1 ds_2 \\ + O(\log Z)$$

$$= J_{31} + J_{32} + O(\log Z), \text{ say.}$$

Then in the integral  $J_{21}$  we shift the line of integration  $I_2$  to  $I_2^*$  defined above. The simple pole at  $s_2 = s_1$  has the residue

$$\frac{1}{2\pi i} \int_{I_1} \frac{\zeta(1+2s_1)^4 W(s_1, s_1, s_1, -s_1)}{\zeta(1+s_1)^6 \zeta(1-s_1) \zeta(1+3s_1)} \cdot \frac{Z^{2s_1}}{s_1^4} ds_1$$

and, shifting  $I_1$  to  $\{s_1; \operatorname{Re}(s_1) = -\eta, |\operatorname{Im}(s_1)| \leq Z^2\}$  we see that this is  $O(Z^{-\eta})$ . On the other hand the triple pole at  $s_2 = -s_1$  has the residue

$$\frac{(\log Z)^2}{4\pi i} \int_{I_1} \frac{\zeta(1+2s_1) \zeta(1-2s_1) W(s_1, -s_1, -s_1, s_1)}{\zeta(1+s_1)^4 \zeta(1-s_1)^4 s_1^4} ds_1 + O(\log Z).$$

Hence we have

$$J_{21} = \frac{(\log Z)^2}{4\pi} \int_{-\infty}^{\infty} \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it, -it, -it, it) t^{-4} dt + O(\log Z).$$

Similarly we have

$$J_{31} = \frac{(\log Z)^2}{4\pi} \int_{-\infty}^{\infty} \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it, -it, it, -it) t^{-4} dt + O(\log Z),$$

$$J_{32} = \frac{(\log Z)^2}{4\pi} \int_{-\infty}^{\infty} \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it, it, -it, -it) t^{-4} dt + O(\log Z).$$

Finally on noting that  $W$  is a symmetric function of its variables and that for real  $t$

$$\begin{aligned} & \frac{|\zeta(1+2it)|^2}{|\zeta(1+it)|^8} W(it, it, -it, -it) \\ &= \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{1}{p} (4|1-p^{it}|^2 - |1-p^{2it}|^2 - 1)\right) \end{aligned}$$

3'

we end the proof of the theorem.

4. Readers are urged to consider more generally the estimation of the sum

$$\sum_{d_1, \dots, d_n} \frac{\mu(d_1)\mu(d_2)\dots\mu(d_n)}{[d_1, d_2, \dots, d_n]}.$$

Addendum

$$C_0 = \frac{3}{2\pi} \int_0^\infty \prod_p \left(1 - \frac{1}{p}\right)^5 \left(1 + \frac{1}{p} (|1 - p^{it}|^4 - 1)\right) \frac{dt}{t^4}$$